Exercise 3.4 *For the DC level in WGN detection problem assume that we wish to have* $P_{FA} = 10^{-4}$ *and* $P_D = 0.99$. *If the SNR is* $10\log_{10} A^2/\sigma^2 = \Leftrightarrow 30\text{dB}$, *determine the necessary number of samples.*

From Equation (3.8) we get that

$$P_D = Q\left(Q^{-1}(P_{FA}) \Leftrightarrow \sqrt{d^2}\right),$$

where $d^2 = NA^2/\sigma^2$. Thus we have

$$\begin{aligned} Q^{-1}(P_D) &= Q^{-1}(P_{FA}) \Leftrightarrow \sqrt{d^2} \\ \Leftrightarrow d^2 &= \left(Q^{-1}(P_{FA}) \Leftrightarrow Q^{-1}(P_D)\right)^2 \end{aligned}$$

and we get

$$\begin{aligned} N &= \frac{\left(Q^{-1}(P_{FA}) \Leftrightarrow Q^{-1}(P_D)\right)^2}{A^2/\sigma^2} \\ &= \frac{\left(Q^{-1}(10^{-4}) \Leftrightarrow Q^{-1}(0.99)\right)^2}{0.001} \\ &= 36546. \end{aligned}$$

Exercise 3.18 *Find the MAP decision rule for*

$$\begin{aligned} \mathcal{H}_0 &: x[0] \sim \mathcal{N}(0,1) \\ \mathcal{H}_1 &: x[0] \sim \mathcal{N}(0,2) \end{aligned}$$

if $P(\mathcal{H}_0) = 1/2$ *and also if* $P(\mathcal{H}_0) = 3/4$. *Display the decision regions in each case and explain.*

According to the MAP decision rule $\mathcal{H}_1$ is decided, if $p(\mathcal{H}_1|\mathbf{x}) > p(\mathcal{H}_0|\mathbf{x})$. Now this is equal to

$$\begin{aligned} p(\mathbf{x}|\mathcal{H}_1)p(\mathcal{H}_1) &> p(\mathbf{x}|\mathcal{H}_0)p(\mathcal{H}_0) \\ \Leftrightarrow \frac{p(\mathbf{x}|\mathcal{H}_1)}{p(\mathbf{x}|\mathcal{H}_0)} &> \frac{p(\mathcal{H}_1)}{p(\mathcal{H}_0)} = \gamma \\ \Leftrightarrow \frac{\frac{1}{\sqrt{2\pi\cdot 2}}e^{-1/(2\cdot 2)x^2[0]}}{\frac{1}{\sqrt{2\pi}}e^{-1/2x^2[0]}} &> \gamma. \end{aligned}$$

Taking a logarithm this yields

$$\begin{aligned} \ln 1/\sqrt{2} \Leftrightarrow 1/4x^2[0] + 1/2x^2[0] &> \ln\gamma \\ \Leftrightarrow 1/4x^2[0] &> \ln\sqrt{2}\gamma \\ \Leftrightarrow |x[0]| &> 2\sqrt{\ln\sqrt{2}\gamma} \end{aligned}$$

For $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 1/2$ we get $\gamma = 1$ and $\mathcal{H}_1$ is decided if $|x[0]| > 1.18$. For $P(\mathcal{H}_0) = 3/4$ and $P(\mathcal{H}_1) = 1/4$ we get $\gamma = 3$ and $\mathcal{H}_1$ is decided if $|x[0]| > 2.40$.

Thus the treshold becomes larger as $P(\mathcal{H}_0)$ increases, which is natural since $\mathcal{H}_0$ is then more likely to occur.