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Optimum multiuser detection

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Abstract

In code division multiple access system K uses share the same channel simultaneously. The separation of users is obtained with unique signature waveforms. The optimum multiuser detection is derived in an additive white Gaussian noise channel both for synchronous and asynchronous transmissions. Noticeable performance gains are obtained compared to the conventional single user detection, but the computational complexity of the optimum detection is too complex to be implemented for example in cellular systems. The demodulation of linear sub-optimal detectors, such as the decorrelating detector, has only linear dependence per demodulated bit, but yet they are able to reduce the multiple access interference quite well.

1 Introduction

The first and most important goal of this presentation is to understand the optimum multiuser detection in CDMA communications. Some discussion is why the multiuser detectors are in general so interesting and how much performance gain is obtained compared to the conventional single-user detectors. Anyhow since in practice the computational complexities of the optimum multiuser detection algorithms are too complex to be implemented, so called sub-optimal detectors have been widely investigated. The second goal is to study the general linear multiuser detectors and compare them to optimum multiuser and conventional single-user detectors. Two example detectors, the decorrelating and the optimal linear detectors, are reviewed and analyzed.

In this paper first the motivation for multiuser detectors is presented in section 1.1, then the

CDMA channel model used is introduced in section 2. The optimum multiuser detection for CDMA signals is presented in section 3. Later the decorrelating detector and the optimum linear multiuser detector are reviewed in section 4. Finally concluding remarks are presented is section 5.

1.1 Why multiuser detection?

The motivation to study the multiuser detectors rises from the weakness of the conventional single user detector. The most essential disability is the lack to investigate the multiple access interference (MAI) caused by other users in the same channel. This reduces the performance of the detector especially when the signal energies are dissimilar, i.e., in the near-far-problem situation. Verdu has shown in [5] that this limitations is not a inherent characteristic of CDMA systems, but just a weak point of the conventional single-user detector.

The optimum multiuser detector outperforms the conventional single user detector at the expense of increase in the computational complexity. Theoretically the complexity of the detector has an exponential dependency on both the number of users and the number of symbols transmitted. With a dynamic programming implementation the exponential dependency on the number of symbols transmitted can be reduced noticeably. Several algorithm have been proposed with varying complexities, for example a Viterbi type algorithm proposed in [5] for asynchronous Gaussian channels is claimed to have only $O(2^K)$ computational complexity, where K is the number of users in the same channel.

Besides the complexity problem the optimum detector has an other disadvantage, that is the requirement of knowledge or estimates of received signal energies. The problem is even more complex when it is kept in mind that the energies can vary during the time. For these two reason a lot a effort has been put into the study of sub-optimal detectors. Examples of these kind of detectors are two the linear receivers presented in section 4. Originally the decorrelating detector and optimum linear detectors were proposed in [9] and [10], where they were stated to be near far resistant with some signal constellation conditions.

2 System model

This section introduces the system model used throughout in this paper. The used notations are mainly based on presentations is papers [5] and [6].

The symbol interval duration is assumed to be equal to T for all users. The set of transmitted signals are marked with A and as general the antipodal set $\{-1, 1\}$ is used unless separately mentioned. A transmitted signal of the kth user in the ith time interval is marked with $b_k(i)$ and total number of users in the channel is K, i.e., $k = 1, \dots, K$. The observed time is divided to the discrete time intervals $t \in [iT, (i+1)T]$ and $2M$ is called the used block size, which indicates the number of transmitted bits or equivalently the number of discrete time interval.

All the transmitted signals, $b_k(i)$, are assumed to be equiprobable and independent. In symbol synchronous transmission each user produces exactly one symbol which interferes with the symbols of other users, see figure 1. Correspondingly in the asynchronous case, as illustrated in figure 2, two symbols from each interferer overlaps the corresponding symbol of the desired user. In symbol synchronous case he information vector received at the ith time-interval is

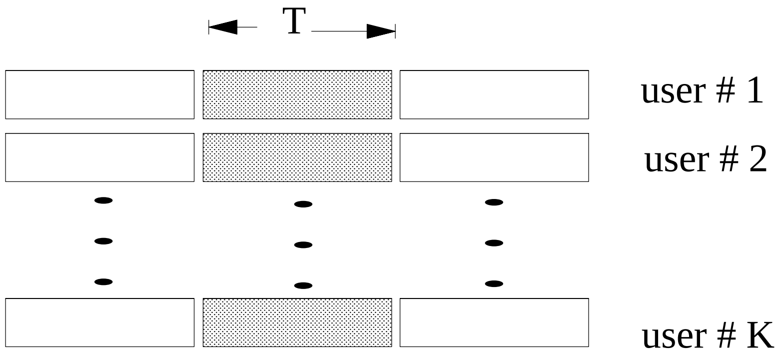

Figure 1: Synchronously transmitted sequence

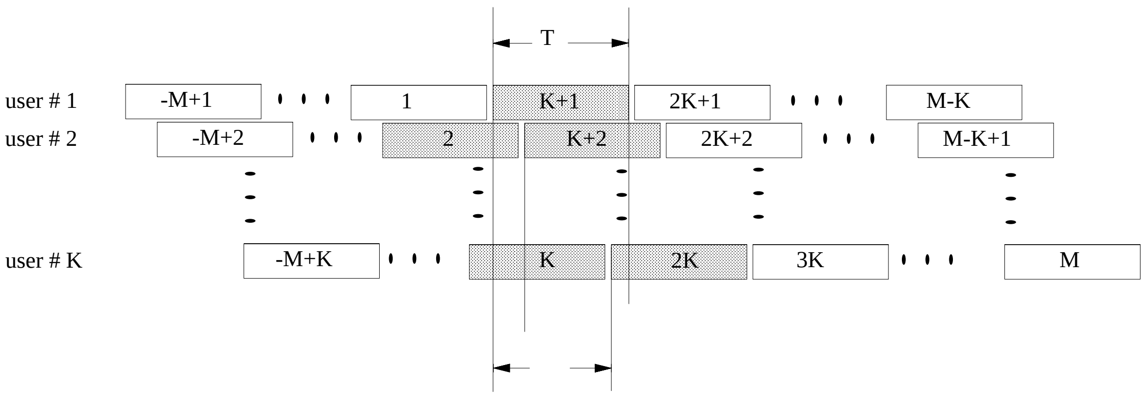

Figure 2: Asynchronous transmission of sequences

$\mathbf{b}(i) = [b_k(i), \ldots, b_K(i)]$. The long information vector $\mathbf{B}$ includes all the bits from all the users, i.e., $\mathbf{B} = [b_1(-M), \ldots, b_K(-M), b_1(-M+1), \ldots, b_K(M)] = [\mathbf{b}(-M), \ldots, \mathbf{b}(M)]$. The numbering of bits is illustrated in figure 2. Using this notations simplies the discussion of detection in the asynchronous transmission.

To obtain the optimum multiuser detector knowledge or estimates of some user specific parameters are required. While the conventional detector needs knowledge only of the signature waveforms, $s_k(t)$ (=0, out size $[0, T]$), the multiuser detector uses also the signal energies, $E(t)$, and in asynchronous case also the transmission delays, τ_k. It is also necessary to obtain the matched filter outputs and cross correlations coefficients before the multiuser detector can be performed. Generally also the signal delays dependent on the time, anyhow this is rarely considered in publications. It is hereafter assumed that this sufficient statistics are available.

The general structure of a multiuser detector is illustrated in figure 3. It consists of a bank of matched filters (one for each user) followed by a decision algorithm common to all users.

The received signal can be expressed with

$$r(t) = S(t, \mathbf{B}) + n(t). \tag{1}$$

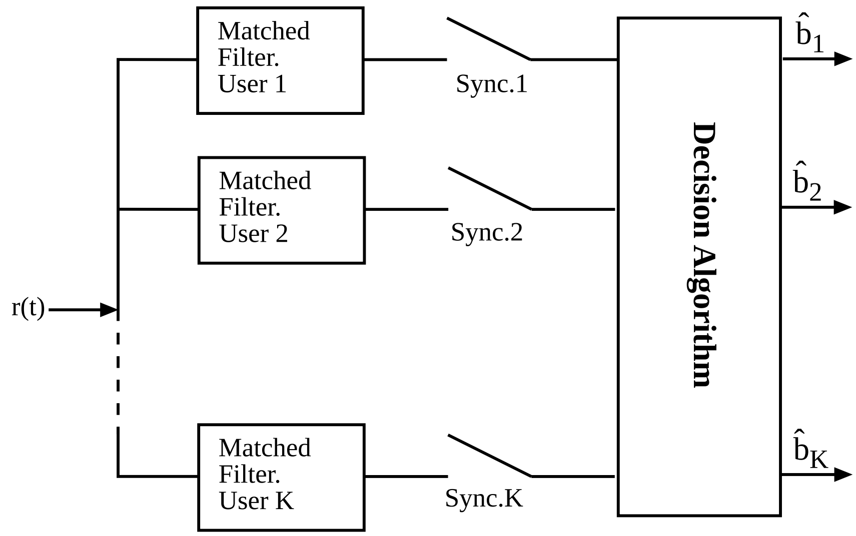

Figure 3: System model for multiuser detectors for asynchronous multiple-access Gaussian channel

where the noise, $n(t)$, is supposed to be additive white Gaussian noise (AWGN) with power spectral density σ^2. In the asynchronous case the

$$S(t, \mathbf{B}) = \sum_{i=-M}^{M} \sum_{k=1}^{K} \sqrt{E_k(t)}\, b_k(i) s_k(t - iT - \tau_k). \tag{2}$$

is the element of $\mathcal{L}_2$ (i.e the Hilbert space of square integrable functions). For the sake of simplicity and without loss of generality the users are numbered such that $0 \leq \tau_1 \leq \ldots \leq \tau_K < T$. In synchronous case all the delays, τ_k, are zero and it is sufficient enough to study separately each time interval i, for example $i = 0$, i.e., $t \in [0, T]$. This one shot approach means that the outer summation (over the time index i) in equation (2) can be dropped out.

When the received signal $r(t)$ is cross-correlated separately with the signature waveform of one user, for example k, the matched filter is achieved for that user. For the user k in time interval i this yields now

$$y_k(i) = \int_{\tau_k+iT}^{\tau_k+iT+T} s_k(t - iT - \tau_k) r(t) dt. \tag{3}$$

The output of a bank of matched filters is marked with $\mathbf{y}$. It is either a vector of size $K \times 1$ or $2MK \times 1$ depending on the synchronization.

3 Optimum multiuser receiver

In this section the optimum CDMA multiuser decision rules for the channel introduced in previous section is presented. The presentation is obtained from [1], [3], [5] and [8]. First some estimation theory is reviewed in section 3.1 and in section 3.2 the optimum detection in symbol-synchronous channel is presented. It is a simple way to gain some appreciation of the general synchronous channel case, which is later reviewed in section 3.3. Finally computational complexity is discussed in section 3.4 and the performance of the detector is analyzed in section 3.5.

3.1 Estimation theory

Due to existence of several users there is no unique optimality criterion. As stated in [5] it is possible to select the optimum sequence in two ways. First choice is to find the set of symbols that maximize the joint *a posteriori* distribution $P[\mathbf{B} \mid \{r(t), t \in \Re\}]$. Here $\Re$ denotes the set of real numbers. Second choice is to select the set that maximizes the marginal *a posteriori* distribution $P[b_k(i) \mid \{r(t), t \in \Re\}]$, $i = -M, \ldots, M, k = 1, \ldots, K$. This selection rule can is be called the locally optimum or minimum-error-probability detection. The later is reviewed for example in [5] for additive-light Poisson multiple-access channels.

Using the Bayes' rule the *a posteriori* distribution can be written

$$P[\mathbf{B} \mid \{r(t), t \in \Re\}] = \frac{P[r(t) \mid \mathbf{B}]P[\mathbf{B}]}{\sum_{\mathbf{B}} P[r(t) \mid \mathbf{B}]P[\mathbf{B}]}. \tag{4}$$

To obtain the maximum of the equation (4) some estimate of the *a priori* density, $P[\mathbf{B}]$, must be established. If this information is not available the the maximum *a posteriori* estimate and the maximum likelihood estimator are identical, see [3]. In CDMA multiuser detection this is the case and therefore the studies can be changed to the maximum likelihood estimation.

It was assumed that all the transmitted sequences of symbols are assumed to be equiprobable, and therefore the equation (4) can be simplified noticeably

$$P[\mathbf{B} \mid \{r(t), t \in \Re\}] = P[\{r(t), t \in \Re\} \mid \mathbf{B}]. \tag{5}$$

To solve this we need to find sequence that maximizes

$$P[\{r(t), t \in \Re\} \mid \mathbf{B}] = C \exp(\Lambda(\mathbf{B})/2\sigma). \tag{6}$$

Here C is a positive scalar independent of $\mathbf{B}$ and the Λ is called the log likelihood function. Since exp is monotonically increasing function it is sufficient and also simpler to find the optimum of this log likelihood function instead of the right hand side of equation (6). For synchronous transmission this optimization is further studied in section 3.2 and for asynchronous case in section 3.3.

3.2 Synchronous Transmission

In the symbol-synchronous channels the symbol epochs of all users coincide at the receiver, see figure 1. In practice this leaves many important classes of CDMA system out and therefore the asynchronous model is referred as a general model. Anyhow the study of symbol-synchronous channels is necessary prerequisite for tackling the the asynchronous channel case and therefore it is presented also here.

The log likelihood function used in equation (6) can in this synchronous case be obtained separately for each time interval i, for example $i = 0$ when $t \in [0, T]$ without losing any information. Thus it is

$$\Lambda(\mathbf{b}) = \int_0^T \left[r(t) - \sum_{k=1}^{K} \sqrt{E_k(0)}\, b_k(0) s_k(t) \right]^2 dt \tag{7}$$

In this one shot approach the signal energies are constants (and positive real numbers, as always) during one time interval. To select the information vector $\mathbf{b}$ that minimizes the $\Lambda(\mathbf{b})$ we need to expand the given integral. This gives us

$$\begin{aligned} \Lambda(\mathbf{b}) &= \int_0^T r(t)^2 dt - 2 \sum_{k=1}^{K} \sqrt{E_k(0)}\, b_k(0) \int_0^T r(t) s_k(t) dt \\ &+ \sum_{j=1}^{K} \sum_{k=1}^{K} \sqrt{E_j(0)}\, \sqrt{E_k(0)}\, b_j(0) b_k(0) \int_0^T s_j(t) s_k(t) dt \end{aligned} \tag{8}$$

Even though this equation is quite complicate, it can be simplified noticeably. First the integral involving $r(t)^2$ is common to all possible sequences $\mathbf{b}(0)$ and is therefore of no relevance determinating which sequence was transmitted and may therefore be neglected. Next comparing the term

$$y_k(0) = \int_0^T r(t) s_k(t) dt, \quad 1 \leq k \leq K, \tag{9}$$

to the equation (3), we see that it is just the matched filter output, y_k, for the user k. The vector of matched filter outputs of all users was denoted with $\mathbf{y} = \{y_k\}$. Finally, the integral

$$\rho_{kj}(0) = \int_0^T s_j(t) s_k(t) dt \tag{10}$$

in the last term of equation (7) is just the cross-correlation of signature waveform of users k and j. The cross-correlation matrix, denote with $\mathbf{R}$, has the elements $\mathbf{R}_{kj} = \{\rho_{kj}\}$ and size $K \times K$.

The optimization problem of equation (7) can now be expressed as a maximization problem in a correlation metrics

$$C(\mathbf{y}, \mathbf{b}) = 2 \sum_{k=1}^{K} \sqrt{E_k(0)}\, b_k(0) y_k - \sum_{j=1}^{K} \sum_{k=1}^{K} \sqrt{E_j(0)}\, \sqrt{E_k(0)}\, b_k(0) b_j(0) \rho_{kj}(0). \tag{11}$$

This has an equal presentation in a matrix from, which is more generally used in literacy. This is:

$$\begin{aligned} C(\mathbf{y},\mathbf{b}) &= 2(\mathbf{eb})^t\mathbf{y} - (\mathbf{eb})^t\mathbf{R}(\mathbf{eb}) \\ &= 2\mathbf{b}^t\mathbf{ey} - \mathbf{b}^t\mathbf{eReb}.. \end{aligned} \tag{12}$$

Where $\mathbf{e}$ is diagonal signal energy matrix with elements $\mathbf{e}_{kk} = \sqrt{E_k}$ and size $K \times K$. Quite often the matrix $\mathbf{e}$ is attached either to the information vector $\mathbf{b}$ or to the cross-correlation matrix $\mathbf{R}$. In this case the $\mathbf{eRe}$ is denoted with $\mathbf{H}$.

Finally the optimum multiuser detection can be formulated in symbol synchronous case as follows

$$\arg \max_{\mathbf{b}\in\{+1,-1\}^K} \left\{2\mathbf{y}^t\mathbf{eb} - \mathbf{b}^t\mathbf{eReb}\right\} \tag{13}$$

In synchronous case if all the users had orthogonal signals the cross correlation matrix $\mathbf{R}$ would be diagonal and therefore the optimum demodulation could be achieved with a bank of matched filters followed by thresholds, i.e., single user detectors. Generally it is not feasible to use orthogonal signals due to the bandwidth limitations and the lack of symbol synchronism among the users. Also the number of available orthogonal signals is restricted (depending on the code-length) and this limits the number of users in a channel.

The task of finding the optimum solution, i.e., the vector $\hat{\mathbf{b}}$, that satisfies the equation (13) depends exponentially on the number of users K, since there exist 2^K different information vectors $\mathbf{b}$. So far nobody has managed to find a polynomial-time algorithm which can solve the optimum multiuser detection even in symbol synchronous case. Problems of this kind of complexity are called nondeterministic polynomial time hard (NP-hard).

3.3 Asynchronous Transmission

The derivation of the optimum solution in the asynchronous case can be done basically similar to the synchronous case, see [1], but the one-shot approach used in symbol synchronous case is no more feasible. As seen in figure 2, there are now exactly two consecutive symbols from each interferer (= the users $j \neq k$) that overlap a desired symbol of user k. The observed time intervals are now $i = -M, \ldots, M$, i.e., $t \in [-MT, MT]$. Theoretically the optimum detection can be applied only after all the signals have been transmitted. The truncation by setting the $2M$ finite may cause some error to the estimate.

An other difference to the synchronous case is that now also the signal energies are time dependent, i.e., $\sqrt{E_k}\,(t)$. The optimum maximum likelihood receiver computes now the log likelihood function

$$\Lambda(\mathbf{b}) \quad = \quad \int_{-Mt}^{MT} \left[r(t) - \sum_{k=1}^{K} \sqrt{E_k(i)}\, b_k(i) s_k(t - iT - \tau_k) \right]^2 dt \tag{14}$$

$$
\begin{aligned}
= & \int_{-Mt}^{MT} r(t)^2 dt - 2\sum_{k=1}^{K}\sqrt{E_k(i)}\sum_{i=-M}^{M} b_k(i)\int_{-Mt}^{MT} r(t)s_k(t-iT-\tau_k)dt \\
+ & \sum_{k=1}^{K}\sum_{j=1}^{K}\sqrt{E_k(i)}\sqrt{E_j(i)}\sum_{i=-M}^{M}\sum_{m=-M}^{M} b_k(i)b_j(m)\int_{-Mt}^{MT} s_k(t-iT-\tau_k)s_j(t-jT-\tau_j)dt
\end{aligned}
$$

The normalized cross-correlations can now be expressed with matrix of size $K \times K$

$$
\mathbf{R}_{kj}(m) = \int_{-\infty}^{\infty} s_k(t-\tau_k)s_j(t+mT+\tau_k-\tau_j)dt \tag{15}
$$

Then, since the modulating signature waveforms are zeros out size the interval $[0, T]$.

$$
\begin{aligned}
\mathbf{R}(m) &= 0,\ \ \forall \mid m \mid > 1, \\
\mathbf{R}(-m) &= \mathbf{R}^t(m)
\end{aligned} \tag{16}
$$

The $\mathbf{R}(1)$ is a upper triangular matrix with zero diagonal, because of the users were numbered according to increasing delays. To make notations more compact lets define th $2MK * 2MK$ symmetric bloc-Toeplitz matrix $\mathcal{R}$ and the diagonal signal energy matrix $\mathbf{E}$ of same size

$$
\mathcal{R} = \begin{pmatrix}
\mathbf{R}(0) & \mathbf{R}(1) & 0 & \dots & \dots & \dots & 0 \\
\mathbf{R}(1) & \mathbf{R}(0) & \mathbf{R}(1) & 0 & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \mathbf{R}(1) & \mathbf{R}(0) & \mathbf{R}(1) \\
0 & 0 & 0 & \dots & 0 & \mathbf{R}(1) & \mathbf{R}(0)
\end{pmatrix} \tag{17}
$$

$$
\begin{aligned}
\mathbf{E} &= \mathrm{diag}([\sqrt{E}_{\,1}(-M), \dots, \sqrt{E}_{\,K}(-M), \sqrt{E}_{\,1}(-M+1), \dots, \sqrt{E}_{\,K}(M)]) \\
&= \mathrm{diag}([\mathbf{e}(-M), \dots, \mathbf{e}(M)])
\end{aligned} \tag{18}
$$

The matched filter output vector $\mathbf{y}$ is also of size $2MK \times 1$ and it can now expressed similar to the synchronous case as

$$
\mathbf{y} = \mathcal{R}\mathbf{E}\mathbf{B} + n(t), \tag{19}
$$

Notice, that the noise vector, $n(t)$, has been correlated and has now the autocorrelation matrix which depends on the cross-correlations

$$
\mathrm{E}[n^t(i)n^t(m)] = \sigma^2\mathbf{R}(i-m). \tag{20}
$$

Here the matrix $\mathbf{R}$ is as defined in equation (15), and is therefore non zero only when $\mid i-j \mid \leq 1$.

With these notations the optimum multiuser detection problem in asynchronous case can be formulates as previously for the synchronous case with the equation (13),

$$\arg \max_{\mathbf{B} \in \{+1,-1\}^{2MK}} \left\{ 2\mathbf{y}^t \mathbf{E} \mathbf{B} - \mathbf{B}^t \mathbf{E} \mathcal{R} \mathbf{E} \mathbf{B} \right\} \tag{21}$$

If the direct approach is used to solve this problem, see [1], selecting the optimum sequences $\hat{\mathbf{B}}$ (of size of $2MK \times 1$) from all the possible 2^{2MK} vectors implies computational complexity exponentially dependent on both number of users K and the block size $2M$. In practice this is too complex to be implemented, especially when K and $2M$ are large. An efficient solution of this combinatorial optimization problem is proposed in [5]. It employs a Viterbi type algorithm (see [1] or [2]) which has exponential dependency only on the number of users.

According to Verdu [5] the key to the efficient maximization of $\Lambda(\mathbf{B})$ lies in its sequential dependence on the symbols $b_k(i)$. This allows to perform a summation of terms that are dependent only on a few variables at a time. Suppose that a recurrent discrete-time system $x_{i+1} = f_i(x_i, u_i)$, with some initial condition x_{i_0}; a transition-payoff function $\lambda_i(x_i, u_i)$; and a bijection between the set of transmitted sequences and a subset of control sequences $u_i, i = i_0 \ldots, i_f$ (i.e., $\mathbf{B} \leftrightarrow u_i, i = i_0 \ldots, i_f$) can be found such that

$$\Lambda(\mathbf{B}) = \sum_{i=i_0}^{i_f} \lambda(x_i, u_i). \tag{22}$$

Equivalent to the maximization of $\Lambda(\mathbf{B})$ is now a discrete-time deterministic control problem finite input and state spaces and with additive cost. It can therefore be solved by the dynamic programming algorithm either in backward or forward fashion. The optimum decisions cannot be made until all states share a common sub-path, but fortunately a well known advantage of in real-time applications of the forward dynamic programming algorithm (the Viterbi algorithm) is that only a little degradation of the performance occurs when the algorithm uses adequately chosen fixed finite decision delay.

Since there is not a unique way to define the additive decomposition of equation (22), several different algorithms with varying complexities have been proposed. Verdu has obtained an algorithm in [5] that has lower computational complexity than all the other proposals. He did it by fully exploiting the the sequential dependency of the log likelihood function on the transmitted symbols.

3.4 Computational Complexity of the optimum multiuser detector

The implementation of the optimum multiuser detector, or equally a maximum likelihood detector, requires as stated knowledge several parameter values. First the signal waveform are know to receiver and in practice the K-user coherent receiver is assumed to lock the signaling interval and phase of each active user. Then it is possible to obtained internally the cross-correlation coefficients, defined by equation (15) as ρ_{kj}, by cross-correlating the the normalized waveforms with the adequate delays and phases supplied by the synchronization system. Therefore the only requirement beyond the synchronization imposed by the need for the partial cross correlations is the availability of the received signal energies of each user k.

The time required to select the optimum sequence divided by the number of transmited bits as $M \to \infty$ is called the *time complexity per binary decision* (TCB). The computational complexity of the single-user detector and the optimum multiuser detector is noticeable different. For the single-user detector TCB is independent of the number of users. It is assumed that all the users employ binary modulation. For the synchronous optimum multiuser detection the complexity is simply derived. To select the optimum sequence of size $K \times 1$ from all the possible 2^K possible choices is a NP hard problem with computational complexity $O(2^K)$.

In asynchronous case the derivation of the computational complexity is more difficult task. The direct approach is to select the optimum vector of size $2MK \times 1$ from all the possible sequences 2^{2MK}. It is possible to reduce the complexity by the choice of used algorithm. The algorithm based on decomposition of equation (22) and defined in [5] is stated to have dimensionality of the state space equal to 2^{K-1} and since each stage is connected to two states in the previous stage, resulting in TCB $= O(2^K)$. The multiuser detector that minimizes the probability of error is stated to have same structure, but instead of forward Viterbi algorithm it uses backward-forward dynamic programming. The computational time complexity is also compared against the some earlier defined decision algorithms and it is stated to be the lowest of all.

3.5 Performance of the optimum multiuser detector

The *efficiency* is defined as the ratio between the signal to noise (SNR) ratio required to achieve the same uncoded bit error rate in the absence of interfering users and the actual SNR. The limit as the background Gaussian noise level goes to zero ($\sigma \to 0$), the *asymtotic efficiency* , characterizes the performance loss when the bit errors are due to interfering users rather than to channel noise. The asymptotic efficiency, η_k is a real number in the interval $[0, 1]$ and if the the provability of bit error, P_k, is nonzero in the absence of the background noise then $\eta_k = 0$. Correspondingly the nonzero η_k implies that $P_k \to 0$ as $\sigma \to 0$.

The computation of *asymtotic efficiency* for asynchronous optimum multiuser detection is also a NP-hard problem, while for the single user detector it is the quite simple to calculate. In [9] it is given as:

$$\eta_k^c = \max^2 \left\{ 0, 1 - \sum_{j \neq k} \mid \mathbf{R}_{jk} \mid \frac{\sqrt{E_j}}{\sqrt{E_k}} \right\} \tag{23}$$

The kth user detector is near far resistant only when all the cross-correlations $\mathbf{R}_{jk} = 0, \;\; \forall j \neq k$. Otherwise the the greatest lowest bound (infimum) of asymptotic efficiency is 0.

In the optimum multiuser detection the kth user error probability is asymptotically equivalent to that of a binary test between the two closest hypotheses that differ in the kth bit. It can be therefore formulated as follows:

$$\eta_k = \frac{1}{\sqrt{E_k}} \min_{\epsilon \in \{-1,1\}^K \text{and } \epsilon_k = 1} \epsilon^t \mathbf{H} \epsilon \tag{24}$$

where the ϵ is the error sequence which corresponds to the difference between $\mathbf{b}$ and the the sequence selected by the detector. The channel matrix was in synchronous case $\mathbf{H} = \mathbf{eRe}$. In the two user symbol synchronous case this asymptotic efficiency is

$$\eta_k = \min \left\{ 1, 1 + \frac{E_k}{E_j} - 2 \mid \rho_{12} \mid \frac{\sqrt{E_k}}{\sqrt{E_j}} \right\}, \tag{25}$$

where $\{k, j\} \in \{\{1, 2\}, \{, 2, 1\}\}$. In [9] figure 3 illustrates the differences in asymptotic efficiency in a two user case for single user, optimum multiuser and decorrelating detector.

Even thought no explicit form of asymptotic efficiency can be obtained to general case a closed form is studied in [9] for the synchronous case. Optimum detection outperforms the conventional single-user detector with this measurement noticeable.

In [5] it is shown that the minimum multiuser error probability is equivalent in the low noise region to that of single user detection with reduced power. Several numerical examples of the performance gain obtained with respect to bit error probability of the optimum multiuser detection have been given in various articles.

4 Linear detectors

It is known that no detector, linear or nonlinear, can outperform the optimum detector with respect to near-far resistance, but the complexity of the optimum detector has been a good motivation to intensive study of various sub-optimal solutions. The goal is to achieve the performance of the optimum receiver with less computational complexity.

In this section first the general idea of linear detectors is derived. Then the decorrelating detector is presented and finally the optimum linear detector is review. More details about the these detectors can be found in [9] and [10].

As define in [10] a linear detector for a bit i of user k is characterized by a vector $\nu^{k,i} \in L$, where L is the vector space with elements of size $2MK \times 1$. Each vector consists of real numbers, i.e., $\nu_j^{k,i}(m) \in \Re$, for further study about the linear spaces see [4]. The numbering of the elements is performed similarly to the numbering of the long information vector $\mathbf{B}$, The unit vector $\mathbf{u}^{k,i} \in L$ has components $\mathbf{u}_j^{k,i}(m) = \delta_{kj}\delta_{mi}$, where δ is the Kronecker delta function. Note that k and j refer to user and i and m to the time intervals.

The purpose is to maximize the *asymtotic efficiency* , η_k, (or equally minimize the probability of bit error, P_k, in the low-noise region). This was defined in section 3.5 as the performance characteristics that indicates the loss due to the existence of other users in the channel. In the symbol synchronous case this efficiency for linear detectors was defined in [9] as

$$\eta_k(\mathbf{Q}) = \max^2 \left\{ 0, \frac{1}{\sqrt{E_k}} \frac{(\mathbf{QH})_{kk} - \sum_{j \neq k} \mid (\mathbf{QH})_{kj} \mid}{\sqrt{(\mathbf{QHQ}^t)_{kk}}} \right\} \tag{26}$$

where $\mathbf{H}$ was the channel matrix, $\mathbf{eRe}$ and $\mathbf{Q}$ the linear detector matrix with rows $\nu^{k,i}$ (of size $1 \times K$).

In [9] it was shown that the generalized inverse detectors achieves the same degree of near far resistance as the optimum detector.

The optimization task can be formulated in the general asynchronous case as follows: which linear mapping $\mathbf{Q} : \Re^{2MK} \to \Re^{2MK}$ maximizes the *asymtotic efficiency* of the decision scheme

$$\hat{\mathbf{B}} = \mathrm{sgn}(\mathbf{Qy}) = \mathrm{sgn}(\mathbf{Q}\mathcal{R}\mathbf{EB} + \mathbf{Q}n), \tag{27}$$

The matrix $\mathbf{Q}$, the linear detector matrix of size $2MK \times 2MK$ has row vectors $\nu^{k,i}$.

4.1 The Decorrelating Detector

The decorrelating detector reviewed in this section is a solution to the generalized likelihood ratio test or maximum likelihood detector when the signal energies are not known at the receiver. It is a linear multiuser detector which recovers the transmitted bits without multiuser interference in the hypothetical case of no background noise. Yet the decorrelating detector is not very close to optimum especially if the signal energies are different.

Again it is easier to consider the synchronous case to understand the nature of the decorrelating detector. The linear memoryless transformation of the decorrelating detector is performed by a generalized inverse of the signature cross-correlation matrix $\mathbf{R}$. Recall the matched filter equation (19)

$$\mathbf{y} = \mathbf{Reb} + n \tag{28}$$

in which the noise has auto covariance matrix $\mathbf{R}$.

To illustrate the decorrelating detector we draw the decision regions in the two user case. In the domain $\mathbf{R}^{-1/2}\mathbf{y}$ where the noise (of the matched filter) is spherically symmetric and Gaussian the decision regions are given by perpendicular bisections of the segments between the different hypotheses denoted with A,B,C and D. In three (or more) dimensional case the decision region are cones with vertices at origin. The mapping with the decorrelating detector matrix transforms the matched filter output to one quadrants. The final selection is performed then with the sgn function.

Since no knowledge or estimate of the signal energies is available, except that they are positive, the best linear solution can be achieved with simple matrix calculation. The problem size is $K \times 1 = (K \times K) * (K \times K) * (K \times 1)$ and can be solved for $\mathbf{eb}$ assuming that the cross-correlation matrix $\mathbf{R}$ is invertible. This is true if only if the signature waveforms are linearly independent of each other. The decorrelating detector is defined as follows:

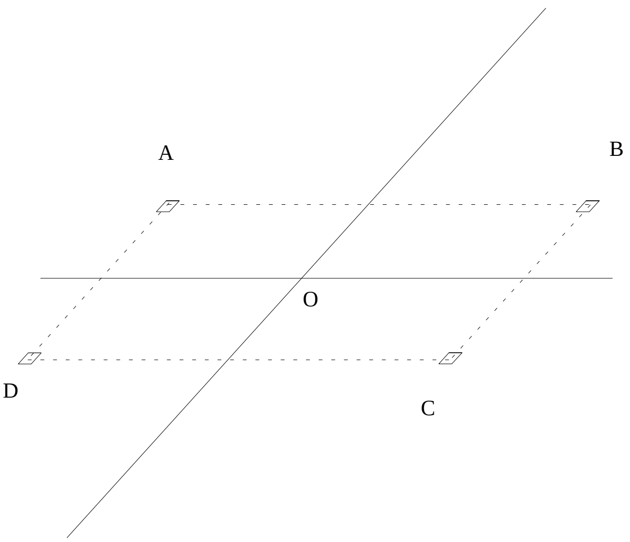

Figure 4: Hypotheses and decision quadrants in two user case for decorrelating detector.

$$\hat{\mathbf{b}} = \mathrm{sgn}(\mathbf{R}^{-1}\mathbf{y}). \tag{29}$$

To verify that the detector really eliminates the multiuser interference lets write the previous equation with an unrealistic assumption that no noise is present in the channel

$$\mathrm{sgn}(\mathbf{R}^{-1}\mathbf{y}) = \mathrm{sgn}(\mathbf{R}^{-1}\mathbf{R}\mathbf{e}\mathbf{b}) = \mathrm{sgn}(\mathbf{e}\mathbf{b}). \tag{30}$$

This is an unbiased estimate of $\mathbf{b}$. It means that the decorrealating detector is invariant for each bit, $b_k(i)$, with respect to received energies. The name *decorrelating* originates from this important property of the detector.

Notice that also the bit error rates P_k are independent of energies of interfering users and the asymptotic efficiency η_k, defined in equation (26) does not depend on any signal energies.

In the asynchronous case for the finite block size $2M$ the decorrelating detector, $\mathbf{q}^{k,i}$, is a linear detector for which

$$\mathcal{R}\mathbf{q}^{k,i} = \mathbf{u}^{k,i}. \tag{31}$$

The existence of the solution is proven in [10]. And the synchronous case, equation (29) is just a simplified form of this equation.

4.1.1 Performance and Complexity

The decorrelating detector offers substantial improvement in asymptotic efficiency compared to conventional single-user detection and has the same near-far resistance as the optimum detector with some limitations on signal energies and cross-correlations. It has been described and analyzed in various articles, from which one of the earliest is [9]. Later it was generalized for asynchronous channels in [10].

The decorrelating detector has linear complexity in the number of users, since it is obtained by performing a linear transformation of the matched filter output. An efficient method for obtaining the decorrelating detector is the the square-root factorization method described in appendix of [1]. The main disadvantage of the detector is computation required to calculate the decorrelating coefficients, i.e., inverting the cross-correlation matrix.

4.2 The optimum linear receiver

For simplicity lets consider the symbol synchronous case of equation (27) which defined the general linear multiuser detection problem. To estimate one user one bit the equation is as follows:

$$\hat{b}_k(i) = \mathrm{sgn}(\nu^{k,i}\mathbf{y}) = \mathrm{sgn}(\nu^{k,i}\mathbf{Reb} + \nu^k(i)n), \tag{32}$$

where the $\nu^{k,i}$ denotes the linear detector for user k in the i time interval, which now is of size $K \times 1$. This optimization problem can be interpreted in terms of decision regions. We need to find the optimum partition of the K-dimensional hypotheses space into K decision cones with vertices at the origin. The surfaces of these cones determine the column of the inverse $\mathbf{Q}^{-1}$, which is the linear mapping we are trying to find. If the $\mathbf{Q}$ is applied to the cone configuration cones are mapped to quadrants. In the two user decorrelating detector case the decision regions are seen in figure 4. In general linear detector case the hypotheses (A, B, C, D) The final output of the detector is obtained by evaluating the sgn of each dimension.

The user kth user optimal linear transformation $\mathbf{Q}_k(\mathbf{y}) = (\nu^{k,i})^t\mathbf{y}$ can formulated as follows:

$$\begin{aligned}(\nu^{k,i})^t &= [1; \mathrm{sgn}\rho_{12} \min\{1, \mid \rho_{12}(\frac{\sqrt{E}_k}{\sqrt{E}_j})\}] & (33)\\ &= \begin{cases} [1; -\mathrm{sgn}\rho_{12}], & \text{if} \frac{\sqrt{E}_k}{\sqrt{E}_j} \leq \mid \rho_{12} \mid \\ \mathbf{b}_k^t, & \text{otherwise} \end{cases} & (34)\end{aligned}$$

Here either $k = 1$ and $j = 2$ or vice versa. The lower equation implies the decorrelating detector. In the range $\frac{\sqrt{E_j}}{\sqrt{E_k}} < \rho$ the asymptotic efficiency of the two user case equals to one defined for optimum detector in equation (25). Otherwise the optimum linear detector for user k is equal to the decorrelating detector, since $\mathbf{b}_k^t$ is just the kth row of decorrelating detector.

Determination of the K user case of the optimum linear detector leads to a situation were we have G ($0 \leq G \leq K$) equations with G unknown parameters. An algorithm proposed for synchronous case in [9], but it hasn't been studied in great deals in literature. The decorrelating detector is far more known and analyzed and this optimum linear detector.

5 Conclusion

This paper has reviewed the optimum multiuser detection in CDMA systems. The motivation to study multiuser detectors was based on the weaknesses of the conventional single user detector. The main problem of the conventional detection is the lack of competence to recover reliably the symbols of the weak users even in situation with low cross-correlations of the signature waveforms.

The optimum detection outperforms the conventional detector in performance clearly, since it is able to subtract the multiple access interference (MAI) for each user. In the same time the complexity of the algorithm is considerable changed. With careful design of the receiver algorithm the time complexity per bit (TCB) can be reduced from $O(2^{2MK})$ to $O(2^K)$ in the binary modulation, where K is the number of users. Especially in the situations were users have very different signal energies or strong cross-correlations the difference in performance is noticeable. In the absence of noise the optimum multiuser detector can remove the interference of other users what ever the signal energies are, if the signature waveforms are not linearly dependent. This feature is called near far resistance.

The sub-optimal detectors which have computational complexity only linearly dependent on the number of users K are called the linear multiuser detectors. They have shown to be able to achieve performance gains over the conventional receiver and with some limiting conditions on signals constellations are near far resistant, i.e., when the background noise level is reduced the bit error rate goes to zero. They can exploit the structure of the multiple access interference caused by other users in the channel.

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